On topological fluid mechanics of non-ideal systems and virtual frozen-in dynamics

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Euler and Navier-Stokes have variant systems with dynamical invariance of helicity and thus (weak) topological equivalence, allowing a strong 'frozen-in' (to, or, dually, 'Lie-carried' by the *virtual* velocity $\boldsymbol{V}$) formulation of the vorticity with a flavor of 'inverse Helmholtz theorem'. We remark on the non-ideal (statistical) topological fluid mechanics (TFM) for (1) the Constantin-Iyer formulation of Navier-Stokes, (2) our own extension of the Gallavotti-Cohen type dynamical ensembles of modified Navier-Stokes with energy-helicity constraints and (3) the Galerkin truncated Euler, as the typical case variants with dynamical time reversibility and helicity invariance. Ideal TFM is thus bridged with non-ideal flows. An example virtual (Lie-)carrier of the vorticity in a Galerkin-truncated Euler system is calculated to demonstrate the issue of determining $\boldsymbol{V}$.

1. Introduction

When some object such as the (helicity) density or vorticity of a flow is not ideally carried by the same fluid, subjected to mixed effects of, say, pressure, diffusion, external stirring and advection on its movement, its identification may be a problem. It is then interesting to specify the 'virtual' carrier(s) of the geometrical object for a *topological fluid mechanics* (TFM) description of the non-ideal situation.

Traditional TFM is based on the Helmholtz theorems which again on the 'frozen-in' [to the flow $\boldsymbol{r}(t)$ with velocity $\boldsymbol{u} = \dot{\boldsymbol{r}}$] property of the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, or, dually, 'Lie-carrying' (Arnold & Khesin 1998, not very essential for this work but helpful for some clarification in Sec. 3.3.2) of the 2-form vorticity (flux)

$$\partial_t \boldsymbol{\omega}(\boldsymbol{r},t) = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) \ \cancel{+\boldsymbol{D}+\boldsymbol{S}}, \text{ or, dually, } \partial_t \mathtt{w} + L_{\boldsymbol{u}}\mathtt{w} = \cancel{\mathtt{D}+\mathtt{s}} \text{ with } \mathtt{w}^{(2)} = \mathtt{du}^{(1)}, \quad (1.1)$$

where the internal damping term $\boldsymbol{D}$ ($= \nu\nabla^2\boldsymbol{\omega}$ for Newtonian fluids) and external source $\boldsymbol{S}$ do not act: These two terms can both vanish, which however is a special case of their *balancing each other in detail/space and time* (as indicated by the slashes across them), as in the case of incompressible Euler equations whose regular solutions conserve both the kinetic energy and helicity, *dynamically*, i.e., with respect to each realization. Lagrangian 'frozen-in' indicates some topological equivalence and the 'Cauchy formula' follows, in the fashion of Constantin & Iyer (2008)'s stochastic Weber formulation of Navier-Stokes,

$$\boldsymbol{\omega}(\boldsymbol{r},t) = [(\nabla\boldsymbol{r})\boldsymbol{\omega}_0] \circ \boldsymbol{R}, \text{ with } \boldsymbol{\omega}_0 = \boldsymbol{\omega}(\boldsymbol{R},t_0) \text{ and the inverse } \boldsymbol{R} = \boldsymbol{r}^{-1} = \boldsymbol{r}(\boldsymbol{R},t_0). \quad (1.2)$$

Given the topological interpretation (Moffatt 1969) of helicity

$$\mathcal{H}(t) = \int_{\mathcal{D}} \boldsymbol{\omega}\cdot\boldsymbol{u}\, d^3\boldsymbol{r}, \quad (1.3)$$

double application of the Helmholtz-Kelvin theorems by taking a simple material loop

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to be itself a vortex line implies the Lagrangian conservation law $D\mathcal{H}/Dt = 0$, and that, with appropriate boundary conditions to have vanishing or balanced flux across the boundaries, the (Eulerian) conservation law $d\mathcal{H}/dt = 0$ with fixed $\mathcal{D}$. When the vorticity is curdled on closed (oriented) vortex filaments $\boldsymbol{C}_n$ with flux strength κ_n, $n = 1, 2, ...$, the Biot-Savart law transforms Eq. (1.3) with $\boldsymbol{\omega} \cdot \boldsymbol{u} = \sum_i \kappa_i \delta(\boldsymbol{r} - \boldsymbol{C}_i)$ into

$$\mathcal{H} = \sum_{i,j} \frac{\kappa_i \kappa_j}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\boldsymbol{r}_i \times d\boldsymbol{r}_j \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} = 2 \sum_{i \leqslant j} l_{ij} \kappa_i \kappa_j, \tag{1.4}$$

where $l_{ij} = l_{ji}$ between C_i and C_j is the Gauss linking integral (Moffatt & Ricca 1992, with $i = j$ indicating the limit of self-linking). At different moments of the smooth Euler flow the frozen-in vortex link is *strongly equivalent*, and for $\mathcal{H} \neq 0$, reversible but not amphichiral. For non-ideal flows, to be discussed below, with fixed helicity, the topology is invariant in a *weak* sense, which will be made strong with a *virtual* carrier $\boldsymbol{V}$. The field variable v, vector or scalar, may be represented with orthonormal bases $\{\phi_n\}$

$$v = \sum_n \hat{v}_n \phi_n, \tag{1.5}$$

called the generalized Fourier expansion or transformation for continuous case. For some analytical calculations or in computer simulations, only finite n is kept, say, bounded by some (large) value N, resulting in Galerkin-truncated dynamics for $v_{\mathrm{T}} = \sum_{n<N} \hat{v}_n \phi_n$. In such a case, the dynamical equation corresponding to the original frozen-in one is

$$\partial_t \boldsymbol{\omega}_{\mathrm{T}} = [\nabla \times (\boldsymbol{u}_{\mathrm{T}} \times \boldsymbol{\omega}_{\mathrm{T}})]_{\mathrm{T}} = \nabla \times (\boldsymbol{u}_{\mathrm{T}} \times \boldsymbol{\omega}_{\mathrm{T}})_{\mathrm{T}}. \tag{1.6}$$

The original frozen-in property is in general also 'truncated' out. There is however a steady state with $\partial_t \boldsymbol{\omega}_{\mathrm{T}} = \partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{u}_{\mathrm{T}} \times \boldsymbol{\omega}_{\mathrm{T}})_{\mathrm{T}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = 0$. By steadiness, the truncation is kept by the flow itself, so basically all the discussions of 'Topology of Steady Fluid Flows' (Chap. II of Arnold & Khesin 1998) apply here.

The truncation may be viewed as the effects of some external forcing exactly canceling the excitations or of infinite damping rate on those truncated modes. Note that Eq. (1.6) is *dynamically time-reversible* in the sense that the equation is not changed by reversing the time $t \to -t$ [and simultaneously reversing the directions/signs of the velocity and vorticity which involve the time derivatives of the (unchanged) spacial coordinates], while, under appropriate conditions such as ergodicity, the *statistical arrow of time* points to the absolute equilibrium state (Kraichnan & Montgomery 1981). The behavior of the Galerkin truncated Euler equations have long been analyzed and simulated (Kraichnan & Montgomery 1981; Krstulovic et al. 2009; Zhu, Yang & Zhu 2014).

For the explicitly driven-dissipative system, with $\boldsymbol{D}$ and $\boldsymbol{S}$ not balanced in detail,

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \boldsymbol{D} + \boldsymbol{S}. \tag{1.7}$$

Of course, as mentioned in the last paragraph Eq. (1.6) may actually be viewed as a result of some particular choice of $\boldsymbol{D}$ and $\boldsymbol{S}$. In particular, $\boldsymbol{D}$ may be so chosen that Eq. (1.7) is still dynamically time reversible and respects some physical constraints, such as the invariance of energy dynamically, i.e., fixed in time (She & Jackson 1993; Gallavotti & Cohen 1995). Actually, Gallavotti (2014) and colleagues have established and tested a so-called 'equivalent ensemble' theory for nonequilibrium dynamics, in particular turbulence, involving the chaotic/Anosov hypothesis, fluctuation theorem and large deviation estimation law for the Sinai-Ruelle-Bowen (SRB) distribution and equivalent ensemble conjecture. Though the theory seems to have been reasonably checked in other different dynamical systems (Gallavotti 2014), the three-dimensional

Navier-Stokes problem has not been well investigated, except for the work of She & Jackson (1993) with however much stronger energy constraints (on each wavenumber shell); and, helicity has never been applicable to constitute the dynamical constraint (which will be accomplished in this note.)The Constantin-Iyer formulation, the Galerkin truncated Euler and the Gallavotti-Cohen modification of Navier-Stokes share important features relevant to the TFM of the ideal Euler equation, so it is our purpose to make the connections tighter by studying the virtual frozen-in dynamics, emphasizing the helicity role. Some major results are put down as Lemma, Theorem or Corollary forms, but should be simple enough for general fluid mechanics readers.

2. Dynamical invariance of helicity and virtual frozen-in formulation

2.1. *Dynamical invariance of the kinetic energy and helicity*

We exploit the helical representation, which is intrinsic, applicable for any flow domain $\mathcal{D}$ and which corresponds to the expansion of $\boldsymbol{v}$ into eigenmodes of the nondimensionalized curl operator $\mathcal{C} = (-\nabla^2)^{-1/2}\nabla\times$ (Moses 1971; Chen, Chen & Eyink 2003, see the latter for a constructive definition of this operator for numerical experiment). We work with incompressible $\boldsymbol{u} = \boldsymbol{u}^+ + \boldsymbol{u}^-$ (the compressible flows can be similarly treated as shown by Moses 1971; Zhu 2016) and can write the generalized Fourier expansion Eq. (1.5) for them as

$$\boldsymbol{u}^c = \sum_n \hat{u}_n^c \boldsymbol{\phi}_n^c,\ \nabla\times\boldsymbol{\phi}_n^c = c\lambda_n\boldsymbol{\phi}_n^c,\ \lambda_n > 0 \tag{2.1}$$

with $c = \pm$ representing the chirality, and by Parseval-Plancherel theorem,

$$\mathcal{E}^\pm = \int_{\mathcal{D}} |\boldsymbol{u}^\pm|^2 d^3\boldsymbol{r}/2 = \sum_n |\hat{u}_n^\pm|^2/2 = \sum_n E_n^\pm, \tag{2.2}$$

$$\mathcal{H}^\pm = \int_{\mathcal{D}} \boldsymbol{\omega}^\pm\cdot\boldsymbol{u}^\pm d^3\boldsymbol{r} = \int_{\mathcal{D}} \pm(-\nabla^2)^{1/2}\boldsymbol{u}^\pm\cdot\boldsymbol{u}^\pm d^3\boldsymbol{r} = \sum_n \pm\lambda_n|\hat{u}_n^\pm|^2 = \sum_n H_n^\pm. \tag{2.3}$$

It follows that their truncated versions are also conserved by the dynamics (1.6) with

$$\text{the kinetic energy } \mathcal{E} = \mathcal{E}^+ + \mathcal{E}^- \text{ and helicity } \mathcal{H} = \mathcal{H}^+ + \mathcal{H}^- \tag{2.4}$$

being ideal invariants (Kraichnan & Montgomery 1981; Waleffe 1992).

It will be shown a bit later that we can also construct $\boldsymbol{D}$ to globally balance $\boldsymbol{S}$, and make Eq. (1.7) also dynamically conserve both $\mathcal{E}$ and $\mathcal{H}$.

2.2. *Virtual frozen-in formulation for dynamical helicity invariance*

We note first that for the invariance of the (generalized) helicity $\frac{d}{dt}\int_{\mathcal{D}}\boldsymbol{\Omega}\cdot\boldsymbol{P}d^3\boldsymbol{r} = 0$, associated with whatever a solenoidal field $\boldsymbol{\Omega} = \nabla\times\boldsymbol{P}$ and its potential vector $\boldsymbol{P}$, $\boldsymbol{\Omega}$ may be regarded 'frozen-in' to some material, *virtual* or real, moving with 'velocity' $\boldsymbol{V}$, filling $\mathcal{D}$ with appropriate boundary conditions,

$$\partial_t\boldsymbol{\Omega} = \nabla\times(\boldsymbol{V}\times\boldsymbol{\Omega}), \tag{2.5}$$

resulting from which is the (pre-assumed) conservation law, with a flavor of 'inverse Helmholtz theorem'. [Such a frozen-in form is common to various plasma fluid models (e.g., Zhu, Yang & Zhu 2014, and references therein) and may also be relevant to potential vorticity dynamics of the atmosphere and the oceans (the latter usually have an extra source terms even without internal damping or external forcing as in, e.g., Gibbon & Holm 2010).] We have

LEMMA 1. With $\boldsymbol{\Omega} = \boldsymbol{\omega}_{(\mathrm{T})}$, the existence and specification issue of the 'inverse Helmholtz theorem' for (1.7) is equivalent to that of $\boldsymbol{V}$ for

$$\nabla \times (\boldsymbol{V} \times \boldsymbol{\omega}_{(\mathrm{T})}) = \nabla \times (\boldsymbol{u}_{(\mathrm{T})} \times \boldsymbol{\omega}_{(\mathrm{T})})_{(\mathrm{T})} + \boldsymbol{D} + \boldsymbol{S}, \tag{2.6}$$

which, with $\nabla \times \boldsymbol{F} = \boldsymbol{D} + \boldsymbol{S}$ for $\boldsymbol{F}$ prescribed in the $\boldsymbol{u}_{\mathrm{T}}$ equation, say, is solved by

$$\boldsymbol{V}_{\perp} = \boldsymbol{\omega}_{(\mathrm{T})} \times \left[(\boldsymbol{u}_{(\mathrm{T})} \times \boldsymbol{\omega}_{(\mathrm{T})})_{(\mathrm{T})} + \boldsymbol{F} + \nabla \psi \right] / |\boldsymbol{\omega}_{(\mathrm{T})}|^2 \text{ with } \boldsymbol{V}_{\parallel} = \boldsymbol{V} - \boldsymbol{V}_{\perp} = \varpi \boldsymbol{\omega}_{(\mathrm{T})}. \tag{2.7}$$

That is, $\boldsymbol{V}$ is determined by $\boldsymbol{V}_{\perp}$ perpendicular to $\boldsymbol{\omega}_{(\mathrm{T})}$, up to a gauge ψ and a component $\boldsymbol{V}_{\parallel}$ parallel to $\boldsymbol{\omega}_{(\mathrm{T})}$; if $\boldsymbol{\omega}_{(\mathrm{T})} = 0$, $\boldsymbol{D} + \boldsymbol{S}$ should also vanish and $\boldsymbol{V}_{\perp}$ can be anything at such singular points.

REMARK 1. Though it is so far non-uniquely determined (we will come back to the issue of caging the freedoms in Sec. 3.3.2), $\boldsymbol{V}$ indeed can be found and set the basis for a non-ideal or non-Euler dynamics with some topological equivalence. Note that the virtual frozen-in dynamics is linear in $\boldsymbol{V}$, and the latter could also be of stochastic nature.

Note that although our discussions are for various appropriate boundary conditions, the periodic boundary condition (PBC) deserves special attentions: It is ideal for investigating the homogeneous turbulence, avoiding the boundary effects, which is widely used in numerical simulations of turbulence and polymer dynamics (Panagiotou, Millett & Lambropoulou 2013). Although the classical topological picture of helicity in a periodic box becomes subtle and can have difficulty, the issue with field lines running out of the periodic box to cause inhomogeneity raised by Berger (1997) may not be a challenge if we are interested in the *statistically homogeneous* problem; and, on the other hand, important topological notions have been worked out for periodic systems (c.f., Panagiotou, Millett & Lambropoulou 2013, and references therein). Thus, our discussions also make sense for problems with standard Fourier expansion of PBC.

3. Topological fluid mechanics for non-ideal systems

3.1. *TFM in the Constantin-Iyer formulation of Navier-Stokes*

There are different virtual carriers of $\boldsymbol{\omega}_{(\mathrm{T})}$ arriving at a point. Assigning each virtual velocity a weight, say, a probability, we are reminiscent of Constantin & Iyer (2008)'s formulation of incompressible Navier-Stokes $\boldsymbol{u} = \mathbf{E}[\tilde{\boldsymbol{u}}]$ with the stochastic Weber formula

$$\tilde{\boldsymbol{u}} = \mathbf{P}[(\nabla^T \boldsymbol{R})(\boldsymbol{u}_0 \circ \boldsymbol{R})] \text{ } (\mathbf{P} \text{ being the Helmholtz/Leray-Hodge projection}) \tag{3.1}$$

for the following 'noisy' trajectories involving the Wiener process $\boldsymbol{W}$

$$d\boldsymbol{r}(\boldsymbol{R}, t) = \boldsymbol{u}(\boldsymbol{r}(\boldsymbol{R}, t))dt + \sqrt{2\nu} d\boldsymbol{W}(t) : \ \boldsymbol{r}(\boldsymbol{R}, t_0) = \boldsymbol{R} \ , \tag{3.2}$$

over which the mathematical expectation $\mathbf{E}$ is taken. Constantin & Iyer (2008) have also proved the stochastic Cauchy formula $\boldsymbol{\omega}(\boldsymbol{r}, t) = \mathbf{E}[\tilde{\boldsymbol{\omega}}(\boldsymbol{r}, t)]$, with $\tilde{\boldsymbol{\omega}}(\boldsymbol{r}, t)$ formally given samely by Eq. (1.2), i.e., 'frozen-in' to the virtual random flow, and proved the stochastic circulation theorem $\oint_C \tilde{\boldsymbol{u}}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \oint_{R(C,t)} \boldsymbol{u}_0 \cdot d\boldsymbol{R}$ with $\boldsymbol{u}_0 = \boldsymbol{u}(\boldsymbol{R}, t_0)$; thus,

$$\oint_C \boldsymbol{u}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \mathbf{E} \oint_{R(C,t)} \boldsymbol{u}_0 \cdot d\boldsymbol{R} = \mathbf{E} \int_{R(S,t)} \boldsymbol{\omega}_0 \cdot d\boldsymbol{S} = \int_{\boldsymbol{S}} \boldsymbol{\omega}(\boldsymbol{r}) \cdot d\boldsymbol{S}. \tag{3.3}$$

It is possible to check (Kuznetsov & Ruban 2000) that $\tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\boldsymbol{u}}$, which, with C taken to be the $\tilde{\boldsymbol{\omega}}$-loop to transform the circulation into helicity, then completes the precise topological picture of their formulation.

We see that the the formulation is *dynamically* time-reversible for each realization, thus the *stochastic Helmholtz-Hankel-Kelvin theorems* and the *stochastic topology conservation*, should not be too surprising. We may put down

COROLLARY 1. *(The Constantin-Iyer virtual (stochastic) frozen-in law.) Constantin-Iyer's stochastic Weber and Cauchy formulae indicate, without explicitly separating the noise out and formally the same as Eq. (1.1),*

$$\partial_t \tilde{\boldsymbol{\omega}} = \nabla \times (\tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{\omega}}). \tag{3.4}$$

We also remark that, as shown by their Remarks (2.4 and 2.8), the above results carries over to the (appropriately) forced case, with the carriers collecting and accumulating the inputs (of momentum and vorticity, say).

Now the *averaged dissipative behavior* or the *statistical time irreversibility* appears to be the same as the established passive scalar relevant mechanisms that the virtual carriers neither collapse onto a single trajectory nor have the same load (Celani et al. 2002, and references therein).

COROLLARY 2. Taking $\tilde{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{u}'$ and $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{\omega}'$, we can decompose Eq. (3.4) like Reynolds for turbulence:

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \mathbf{E}[\nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}')], \text{ i.e., } \nu\nabla^2\boldsymbol{\omega} + \boldsymbol{S} = \mathbf{E}[\nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}')], \tag{3.5}$$

$$\partial_t \boldsymbol{\omega}' = \nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}) + \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}') + \nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}') - \mathbf{E}[\nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}')]. \tag{3.6}$$

REMARK 2. $\mathbf{E}[\nabla \times (\boldsymbol{u}' \times \boldsymbol{\omega}')]$ effectively shows the micro-macro reconnections of the carried four types of vortex links in (3.5 and 3.6), the Navier-Stokes one, in (3.5), of which may however be recovered yet in another sense, as will be presented below.

3.2. *Extending the Gallavotti-Cohen theory and enriching it with TFM*

So far, in constructing the dynamical model for turbulence equivalent ensemble theory (Gallavotti 2014), only one observable, say the energy or enstrophy/dissipation rate is fixed as a constraint. This might be viewed in two ways: it could be believed that one constraint was sufficient to define the turbulence ensemble; the other could that only one appropriate constraint was allowed, for there is only one molecular viscosity coefficient ν in the incompressible Navier-Stokes equation. We will argue that neither of these considerations are necessary or appropriate, and it is reasonable and possible to have an 'equivalent' dynamical ensemble with the energy-helicity dual constraints.

The explanation and discussion can be made with the generalized helical Fourier representation [c.f., Eqs. (2.1,2.2,2.3)] as will be done for the special statistical steady state with all $\mathcal{E}^{\pm}$ and $\mathcal{H}^{\pm}$ fixed, but, we will also demonstrate some results for the general time-reversible chiral dynamical ensembles with only $\mathcal{E}$ and $\mathcal{H}$ fixed, for convenience and simplicity, in the standard Fourier space as an explicit example.

With Eq. (2.1), the modified Navier-Stokes equations read, in the 'two-fluid' style,

$$\partial_t \boldsymbol{\omega}^{\pm} = [\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})]^{\pm} + \mu^{\pm}(t)\nabla^2\boldsymbol{\omega}^{\pm} + \boldsymbol{s}^{\pm}, \tag{3.7}$$

where the chirality projection, refining that of Helmholtz/Leray-Hodge, eliminates the pressure gradient and where we have replaced ν by the chirally designated *dynamical eddy viscosities* $\mu^{\pm}(t)$: This latter goes beyond the conventional treatment (Gallavotti 2014) and allows us to introduce dual constraints. It should be clear enough to work with the general helical expansion (2.1), but we would like to demonstrate it as an example

even more explicitly in the Fourier space, with the replacements $n \to \boldsymbol{k}$ and $\lambda_n \to |\boldsymbol{k}| = k$ in Eqs. (2.1,2.3), by the following theorem.

Theorem 1. *$\mathcal{E}$ and $\mathcal{H}$ in Eq. (3.7) is fixed by*

$$\mu^{\pm}(t) = \pm \sum_{\boldsymbol{k}} \frac{k^2|\hat{u}^{\mp}_{\boldsymbol{k}}|^2 \sum_{\boldsymbol{k}} k(\hat{s}^{+}_{\boldsymbol{k}}\hat{u}^{+*}_{\boldsymbol{k}} - \hat{s}^{-}_{\boldsymbol{k}}\hat{u}^{-*}_{\boldsymbol{k}}) \mp (\hat{s}^{+}_{\boldsymbol{k}}\hat{u}^{+*}_{\boldsymbol{k}} + \hat{s}^{-}_{\boldsymbol{k}}\hat{u}^{-*}_{\boldsymbol{k}}) \sum_{\boldsymbol{k}} k^3|\hat{u}^{\mp}_{\boldsymbol{k}}|^2}{\sum_{\boldsymbol{k}} k^2|\hat{u}^{-}_{\boldsymbol{k}}|^2 \sum_{\boldsymbol{k}} k^3|\hat{u}^{+}_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k}} k^2|\hat{u}^{+}_{\boldsymbol{k}}|^2 \sum_{\boldsymbol{k}} k^3|\hat{u}^{-}_{\boldsymbol{k}}|^2}. \tag{3.8}$$

Proof. Applying Eqs. (3.7, 2.1, 2.2, 2.3 and 2.4), we have

$$d\mathcal{E}/dt = \mu^{-}\sum_{\boldsymbol{k}} k^2|\hat{u}^{-}_{\boldsymbol{k}}|^2 + \mu^{+}\sum_{\boldsymbol{k}} k^2|\hat{u}^{+}_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k}}[\hat{s}^{+}_{\boldsymbol{k}}\hat{u}^{+*}_{\boldsymbol{k}} + \hat{s}^{-}_{\boldsymbol{k}}\hat{u}^{-*}_{\boldsymbol{k}}] = 0, \tag{3.9}$$

$$d\mathcal{H}/dt = \mu^{-}\sum_{\boldsymbol{k}} k^3|\hat{u}^{-}_{\boldsymbol{k}}|^2 - \mu^{+}\sum_{\boldsymbol{k}} k^3|\hat{u}^{+}_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k}} k[\hat{s}^{+}_{\boldsymbol{k}}\hat{u}^{+*}_{\boldsymbol{k}} - \hat{s}^{-}_{\boldsymbol{k}}\hat{u}^{-*}_{\boldsymbol{k}}] = 0, \tag{3.10}$$

which are solved by Eq. (3.8). □

Remark 3. She & Jackson (1993) fixed the one-dimensional energy spectrum $E(k) = E^{+}(k) + E^{-}(k)$ on each k-shell and they also noticed that other models also allow a dynamically time-reversible modifications, which can be similarly extended to include the helicity spectrum $H(k) = H^{+}(k) + H^{-}(k)$ as the constraint duo, by the replacements, say, $\mu^{\pm} \to \mu^{k\pm}_{m}|\boldsymbol{u}^{\pm}|^{2m}$ in Eq. (3.7), and accordingly also $\sum_{\boldsymbol{k}} \to \sum_{|\boldsymbol{k}|=k}$ in (3.9, 3.10). Our emphasis is the intensity-topology or energy-helicity dual aspects for characterizing the statistics of turbulent fluctuation structures, based on which the 'optimal' choice of the detailed model should depend on the purpose and on the characteristics of the problem, such as the forcing mechanisms. For example, we could construct two dynamical eddy viscosities to modify the homochiral Navier-Stokes studied by Biferale, Musacchio & Toschi (2013) [see, also Waleffe (1992), and Zhu, Yang & Zhu (2014)], one emphasizing the (energy dynamics at) larger scales and the other the (helicity dynamics at) smaller ones.] Of course, extra Galerkin truncation may be imposed, to regularize the solution (Gallavotti 2014), say.

Remark 4. $\mathcal{H}^{\pm}$ (and $\mathcal{E}^{\pm}$) may still vary in time, with 'internal' mutual exchange between the $\pm$ chiral sectors to fix $\mathcal{H}$ (and $\mathcal{E}$). So, it appears *not* meaningful to write down the following frozen-in law,

$$\partial_t \boldsymbol{\omega}^{\pm} = \nabla \times (\boldsymbol{V}_{\pm} \times \boldsymbol{\omega}^{\pm}), \tag{3.11}$$

where the index $\pm$ in $\boldsymbol{V}_{\pm}$ do not mean chirality but only that distinct virtual velocities carry the two chiral sectors of $\boldsymbol{\omega}$. But if such a dual frozen-in laws with fixed helicities pairs, somewhat like that of the plasma two-fluid model (e.g., Zhu, Yang & Zhu 2014, and references therein), have some value, it is still possible to achieve it by noting from Eqs. (2.2,2.3) that $\exists\ \lambda^{\pm} > 0$, $\mathcal{H}^{\pm} = \pm\lambda^{\pm}\mathcal{E}^{\pm}$, where $\lambda^{\pm}$ are now taken to be fixed in the special statistical steady state. Now fixing both $\mathcal{H}$ and $\mathcal{E}$ is equivalent to fixing $\mathcal{E}^{\pm}$ or $\mathcal{H}^{\pm}$, by which we can compute $\mu^{\pm}$. Note that such calculated $\mu^{\pm}$ must have $\lambda^{\pm}$ explicitly in the expressions and are consistent with Eq. (3.8) only for specially values of $\lambda^{\pm}$ determined by the setting of the problem (such as $\boldsymbol{s}^{\pm}$, say), but not in general. In other words, only in very special set ups, can we view the flows as pairs of $\boldsymbol{\omega}^{\pm}$ vortexes frozen-in respectively to their own virtual carriers $\boldsymbol{V}_{\pm}$.

So, indeed, we can construct dynamically time reversible modified Navier-Stokes with

fixed $\mathcal{H}$ [or even the one-dimensional helicity spectrum $H(k)$], which allows us to write down the virtual frozen-in law with strong Lagrangian topological invariance (but weak equivalence with the 'real' one), with

COROLLARY 3. *The sum of the two chiral sectors of Eq. (3.7) gives*

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{V} \times \boldsymbol{\omega}) = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + [\mu^+(t)\nabla^2\boldsymbol{\omega}^+ + \mu^-(t)\nabla^2\boldsymbol{\omega}^-] + \boldsymbol{S}. \tag{3.12}$$

REMARK 5. Linearity on $\boldsymbol{V}$ may offer convenience in the analyses with independent statistics of $\boldsymbol{V}$ and that of $\boldsymbol{u}$. That is one may treat Eq. (3.12) as a stochastic partial differential equation with the distribution of $\boldsymbol{V}$ to be specified: This is similar to Constantin & Iyer (2008)'s formulation for Navier-Stokes, but in a 'macroscopic' level.

All the chaotic or Anosov hypothesis, fluctuation theorem and large deviation evaluation with the SRB distribution and equivalent ensemble conjecture (Gallavotti 2014) may carry over, *mutatis mutandis*, to our new model (3.7 and 3.8). The source's injection or extraction (of energy, say) should be balanced by the modified eddy viscosities. Since the source, by its name/definition, will statistically have positive input, the eddy viscosities should statistically be more positive to be damping, and thus the average phase space contraction or the entropy production rate should be positive, indicating the arrow of time for statistical irreversibility. How the system reach the statistical steady state (relevant to the chaotic hypothesis) is not yet clear, but nevertheless the numerical results of She & Jackson (1993, though quite special) indicate the possibility.

3.3. *TFM for the Galerkin-truncated Euler*

3.3.1. *The virtual frozen in law and relevance to Constantin-Iyer*

Letting $\boldsymbol{s}^{\pm} = 0$ in Eq. (3.7) and implementing Galerkin truncation, we have the truncated Euler who dynamically conserves helicity (Sec. 2.1.) We now write down the virtual frozen-in law for it

$$\partial_t \boldsymbol{\omega}_{\mathrm{T}} = \nabla \times (\boldsymbol{V} \times \boldsymbol{\omega}_{\mathrm{T}}). \tag{3.13}$$

REMARK 6. It is natural to require $\boldsymbol{V} \to \boldsymbol{u}$ as $\boldsymbol{u}_{\mathrm{T}} \to \boldsymbol{u}$, so that the virtual frozen-in law can be a 'good' approximation to the ideal one by taking large N. It deserves to mention that by application of the Fourier helical representation (Moses 1971), it is direct to show there are two, and only two (Zhu 2017), *generic* modes that are steady, viz, the *stratified vorticity mode* with uniform velocity in a plane and shear perpendicular to the velocity (vertically sheared horizontal flow) and the *(Gromeka-)Beltrami mode* with velocity co-linear to its vorticity. Then, $\boldsymbol{u}_{\mathrm{T}} = \boldsymbol{u}$ is an example of $\boldsymbol{V}$. Beltrami flow is particularly interesting also because it is shown that the stream/vortex lines of Beltrami flow actually can be of any knot or link type (Enciso & Peralta-Sulas 2012).

REMARK 7. REMARK 5 also applies here. And, Cichowlas et al. (2005) found in their simulations of the Galerkin truncated Euler that the small-scale fluctuations first thermalize, resulting in a dissipative Kolmogorov turbulence at larger scales. Thus, reconnection of vorticity happens at different scale regimes with spontaneous un/re-linking (-knotting). The large-k Fourier modes' thermalization takes energy from the smaller-k modes in such a way similar to the molecules thermal motions of a fluid to whose heat the macroscopic fluid motions are transformed by the viscosity, a way strongly reminiscent of the Constantin-Iyer formulation discussed in the last section.

In other words, the common phenomena of (virtual) frozen-in law and dynamical time reversibility and invariance law of helicity lead us to the analogy between the thermalized modes and the Brownian particles, both of which lead to the breaking down of the Helmholtz theorems for the 'macroscopic' subsystem.

3.3.2. *Two-dimensional-three-component example: 'fictitious transport' and (Lie-)carry*

Extra appropriate physical condition, such as incompressibility, can be imposed on the virtual carrier $\boldsymbol{V}$ of Eq. (3.13), given by Eq. (2.7). For instance, we can choose such $\boldsymbol{V}_{\|}$, with its divergence $\boldsymbol{\omega}\cdot\nabla\varpi$ exactly canceling $\nabla\cdot\boldsymbol{V}_{\perp}$ (with a gauge part $\nabla\times(\boldsymbol{\omega}/|\boldsymbol{\omega}|^2)\cdot\nabla\psi$), that $\nabla\cdot\boldsymbol{V}=0$. Moffatt (2014) presented examples of single-triad flows which are two-dimensional-three-component (2D3C), so we remark on general 2D3C virtual frozen-in issue, with explicit calculations to demonstrate the issue of caging the freedoms in $\boldsymbol{V}$.

For the 2D3C virtual frozen-in flow with $\partial_z=0=k_z$, the equations for ω_z (with the index '$_{\mathrm{T}}$' dropped here) and the incompressibility read

$$\partial_t\omega_z=-V_x\partial_x\omega_z-V_y\partial_y\omega_z+\omega_x\partial_xV_z+\omega_y\partial_yV_z,\ \partial_xV_x+\partial_yV_y=0. \tag{3.14}$$

To calculate $\boldsymbol{V}$, we need the other two horizontal-$h=x,\ y$-component dynamics,

$$\partial_t\omega_h=-V_x\partial_x\omega_h-V_y\partial_y\omega_h+\omega_x\partial_xV_h+\omega_y\partial_yV_h. \tag{3.15}$$

With the freedom of $\boldsymbol{V}_{\|}=\varpi\boldsymbol{\omega}$ and $\boldsymbol{V}_{\perp}$ given by Eq. (2.7), we can first choose such $\varpi(\boldsymbol{r},t)$ that $\varpi\omega_z=\omega_z-V_{\perp z}$, i.e., $V_z=\omega_z$, $V_z=u_z$ or $V_x=u_x$, but the one leading to simplest expressions for the other two components is

$$V_z \text{ satisfying } \partial_t\omega_z=\omega_x\partial_xV_z+\omega_y\partial_yV_z,\ \text{i.e. } V_x\partial_x\omega_z=-V_y\partial_y\omega_z, \tag{3.16}$$

which solves Eq. (3.14, 3.15) by, with each partial derivative '∂' replaced by the index behind a comma for simplicity,

$$\begin{aligned}V_x=&(\omega_{z,y}\omega_{y,t}+\omega_{z,x}\omega_{x,t})\omega_{z,y}/[\omega_{z,x}(\omega_x\omega_{z,yx}+\omega_y\omega_{z,yy}+\omega_{z,x}\omega_{x,y}-\omega_{x,x}\omega_{z,y}\\&+\omega_{z,y}\omega_{y,y})-\omega_{z,y}(\omega_y\omega_{z,xy}-\omega_x\omega_{z,xx}-\omega_{y,x}\omega_{z,y})],\ V_y=-\omega_{z,x}V_x/\omega_{z,y}.\end{aligned} \tag{3.17}$$

REMARK 8. Such $\boldsymbol{V}$, determined by the existing numerical or analytical fields $\boldsymbol{\omega}_{\mathrm{T}}$, could be singular, if indeed, at only some non-generic locations and shows the dramatic effect of Galerkin truncation by its sharp feature of the differences to $\boldsymbol{u}$ or $\boldsymbol{u}_{\mathrm{T}}$. The other choices of V_z mentioned earlier, including even $V_z=0$ (i.e., purely 2D!), also lead to explicit (but generally much more complex) expressions for the other two components of $\boldsymbol{V}$. The usual velocity $\boldsymbol{u}$ should be modified for being transported in the sense of Lagrangian or 'Lie' for the corresponding 1-form (Oseledets 1989), which, of course, by no means indicates the ineligibility of studying scalar-like transport of $\boldsymbol{u}$. Moffatt (2014)'s impossible fictitious transport for ω_z corresponds to dropping the last two terms of (3.15): For the scalar-like transport $\boldsymbol{V}\cdot\nabla\boldsymbol{v}$ of any $\boldsymbol{v}$, the situation depends on the determinant $\det(\partial_jv_i)$, which is generically non-zero for, say, the Galerkin truncated dynamical field and uniquely determines $\boldsymbol{V}$, thus extra condition would lead to overdetermination. $\partial_z=0$ for 2D3C problem already makes the scalar-like transport problem overdetermined (two variables with three equations), and extra incompressibility condition aggravates it.

Part of the above results can be summarized as

THEOREM 2. *Once V_z is specified, the 2D incompressible $\boldsymbol{V}_h$ is in general uniquely determined by the 2D3C field $\boldsymbol{\omega}$, satisfying $\partial_t\boldsymbol{\omega}=\nabla\times(\boldsymbol{V}_h\times\boldsymbol{\omega})$.*

Proof. Taking the gradient of the equation for ω_z in Eq. (3.14), we get yet two, and in

total six, linear algebraic equations to determine the six variables, with specified V_z, of the components of $\boldsymbol{V}_h$ and its Jacobian matrix $\nabla \boldsymbol{V}_h$. □

REMARK 9. Given $V_z = 0$, $\boldsymbol{V}$ is purely 2D; that is, there exists a unique pure 2D carrier. Here, we have prescribed one component of the 2D3C $\boldsymbol{V}$ to by-pass the problem of solving the first-order (spatially) partial-differential equation (PDE), but in general we may cage $\boldsymbol{V}$ by other, probably dynamical, principles. For example, in a fully 3D channel flow, say, appropriate external boundary conditions may be applied, when PDEs like (3.14 and 3.15) are solved; and, internal properties, such as the degree of hydrochirality (Zhu, Yang & Zhu 2014), of $\boldsymbol{V}$ (dynamics) may also be considered.

4. Concluding remarks

An example calculation has demonstrated the issue of determining (the freedoms of) the virtual carrier $\boldsymbol{V}$, with the general considerations being possibly helpful also for other relevant problems, such as the recent efforts to settle down onto a special vortex surface field by introducing particular dynamical constraint, say, Eq. (3.4) in Yang & Pullin (2010). $\boldsymbol{V}$ for each virtual fluid particle in general could be stochastic or have its own dynamics. Since the virtual frozen-in formulation indicates strong topological invariance (beyond helicity), which could actually be the 'real' *wave-like propagation speed* of vorticity or that in some curved space (Besse & Frisch 2017), it may be possible to settle down onto some least-'action' (in the sense of topological complexity or knot 'energy', say) path determined by $\boldsymbol{V}(t)$ by applying path integral techniques using the form in recent derivation of HOMFLYPT polynomials (Liu & Ricca 2015). Application (part of) the spectral constraint as in REMARK 3 may also be considered.

There are infinitely many conservation laws (e.g., Tur & Yanovsky 1993; Cheviakov & Oberlack 2014), but, probably due to the lack of ruggedness with respect to the truncation (Kraichnan & Montgomery 1981), it appears only energy and helicity are of generic dynamical importance for turbulence. However, for the *virtual* frozen-in (or more general Lie-carry) formulation, extra conservation laws might also be used to cage $\boldsymbol{V}$.

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